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On Darboux transformation of the supersymmetric sine-Gordon equation

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Abstract

Darboux transformation is constructed for superfields of the super sine-Gordon equation and the superfields of the associated linear problem. The Darboux transformation is shown to be related to the super Bäcklund transformation and is further used to obtain N super soliton solutions.

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There has been an increasing interest in the study of supersymmetric integrable systems for the last few decades [1–20]. Among the many techniques used to study integrability and to obtain the multisoliton solutions for a given integrable model, the Darboux transformation[2] has been widely used and it has established itself as an economic, convenient and efficient way of generating solutions [21–23]. The Darboux transformation has been employed on some supersymmetric integrable models in recent years [15, 16]. In these investigations multisoliton solutions have been constructed and the ideas are generalized to incorporate the Crum transformation, the Wronskian superdeterminant and Pfaffian-type solutions. The super soliton solutions of the super KdV equation and super sine-Gordon equation have been investigated and it has been shown that the solitons of the KdV and sine-Gordon solitons appear as the body of the super solitons [12–16].

The purpose of this work is to provide a thorough investigation of the Darboux transformation for the super sine-Gordon equation in a systematic way and to obtain the explicit super multisoliton solutions by a Crum-type transformation. Following [17], we write a linear problem in superspace whose compatibility condition is the super sine-Gordon equation. The linear problem then leads to a Lax formalism in superspace. We explicitly write the Darboux transformation for the fermionic and bosonic superfields of the linear system and

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[2] Originally the Darboux transformation was first introduced by Darboux back in 1882, in the study of pseudo-spherical surfaces and later used to generate solutions of the Sturm–Liouville differential equation. The Darboux transformation has now widely been used to generate multisoliton solutions of integrable as well as super integrable evolution equations.

for the scalar superfield of the super sine-Gordon equation. The approach adopted here is different from that adopted in [15]. In [15], the authors have not explicitly constructed the Darboux transformation and N-soliton solutions for the super sine-Gordon equation. We have extended the results of [15] to give N super soliton solutions of the super sine-Gordon equation in terms of known solutions of the linear problem and express it as a series of products of super determinants and fermionic superfields. The Darboux transformation is then shown to be related to the super Bäcklund transformation of the super sine-Gordon equation [5].

We follow the general procedure of writing manifestly the supersymmetric sine-Gordon equation. The equation is defined in two-dimensional super-Minkowski space with bosonic light-cone coordinates[3] $x^{\pm}$ and fermionic coordinates $\theta^{\pm}$, which are Majorana spinors. The superspace Lagrangian density of $N = 1$ super sine-Gordon theory is given by

$$\mathcal{L}(\Phi) = \frac{\mathrm{i}}{2} D_+\Phi D_-\Phi + \cos\Phi, \tag{1}$$

where Φ is a real scalar superfield and $D_\pm$ are covariant superspace derivatives defined as

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\theta^\pm\partial_\pm, \qquad D_\pm^2 = -\mathrm{i}\partial_\pm, \qquad \{D_+, D_-\} = 0,$$

where $\{,\}$ is an anti-commutator. The superfield evolution equation follows from the Lagrangian and is given by

$$D_+D_-\Phi = \mathrm{i}\sin\Phi. \tag{2}$$

Equation (2) is invariant under $N = 1$ supersymmetry transformations.

Let us first recall that the super sine-Gordon equation appears as the compatibility condition of the following linear system of equations [17]:

$$D_\pm\Psi = \mathcal{A}_\pm\Psi, \tag{3}$$

where the superfield Ψ is expressed in terms of bosonic and fermionic superfield components as

$$\Psi = \begin{pmatrix} \psi \\ \phi \\ \chi \end{pmatrix},$$

where ψ, ϕ are bosonic superfields and χ is a fermionic superfield and $\mathcal{A}_\pm$ are 3×3 matrices given by

$$\mathcal{A}_+ = \frac{1}{2\sqrt{\lambda}} \begin{pmatrix} 0 & 0 & \mathrm{i}\,\mathrm{e}^{\mathrm{i}\Phi} \\ 0 & 0 & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\Phi} \\ -\mathrm{e}^{-\mathrm{i}\Phi} & \mathrm{e}^{\mathrm{i}\Phi} & 0 \end{pmatrix},$$

$$\mathcal{A}_- = \sqrt{\lambda} \begin{pmatrix} \frac{\mathrm{i}D_-\Phi}{\sqrt{\lambda}} & 0 & -\mathrm{i} \\ 0 & -\frac{\mathrm{i}D_-\Phi}{\sqrt{\lambda}} & \mathrm{i} \\ -1 & 1 & 0 \end{pmatrix}.$$

The zero-curvature condition in superspace

$$D_+\mathcal{A}_- + D_-\mathcal{A}_+ - \{\mathcal{A}_+, \mathcal{A}_-\} = 0, \tag{4}$$

gives equation (2). The zero-curvature condition (4) is essentially the compatibility condition of the linear system (3). As in the bosonic case, the zero-curvature condition of the super sine-Gordon equation gives a Lax representation; one can directly find the bosonic Lax operator

[3] Our spacetime conventions are such that the orthonormal and light-cone coordinates are related by $x^\pm = \frac{1}{2}(t \pm x)$ and $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$.

for the supersymmetric case such that the given operators obey the Lax equation. The Lax equation gives the evolution of the spectral problem and solves the super sine-Gordon equation in the spirit of the inverse scattering method[4] [17].

The Darboux transformation is used to generate multisoliton solutions of integrable as well as super integrable evolution equations. Given a superfield equation (2) and its associated linear system (3), one can construct a Darboux transformation on the superfields ψ, ϕ and χ of the linear system (3) and on the sine-Gordon superfield Φ such that the new transformed fields obey the same differential equations for the value of the spectral parameter $\lambda = \lambda_0$. The one-fold Darboux transformation for the system (3) is given by

$$\begin{aligned}
\psi(\lambda_0) &\to \psi[1](\lambda_0;\lambda_1) = \lambda_0\phi(\lambda_0) - \lambda_1\frac{\phi_1(\lambda_1)}{\psi_1(\lambda_1)}\psi(\lambda_0) - \mathrm{i}\sqrt{\lambda_0\lambda_1}\frac{\chi_1(\lambda_1)}{\psi_1(\lambda_1)}\chi(\lambda_0),\\
\phi(\lambda_0) &\to \phi[1](\lambda_0;\lambda_1) = \lambda_0\psi(\lambda_0) - \lambda_1\frac{\psi_1(\lambda_1)}{\phi_1(\lambda_1)}\phi(\lambda_0) - \mathrm{i}\sqrt{\lambda_0\lambda_1}\frac{\chi_1(\lambda_1)}{\phi_1(\lambda_1)}\chi(\lambda_0),\\
\chi(\lambda_0) &\to \chi[1](\lambda_0;\lambda_1) = -(\lambda_0+\lambda_1)\chi(\lambda_0) + \sqrt{\lambda_0\lambda_1}\frac{\chi_1(\lambda_1)}{\psi_1(\lambda_1)}\psi(\lambda_0) + \sqrt{\lambda_0\lambda_1}\frac{\chi_1(\lambda_1)}{\phi_1(\lambda_1)}\phi(\lambda_0),
\end{aligned}\tag{5}$$

where $\psi(\lambda_0), \phi(\lambda_0)$ and $\chi(\lambda_0)$ are the solutions of the system (3) with $\lambda = \lambda_0$ and $\psi_1(\lambda_1), \phi_1(\lambda_1)$ and $\chi_1(\lambda_1)$ are the solution of the system (3) with $\lambda = \lambda_1$. The functions $\psi[1](\lambda_0;\lambda_1), \phi[1](\lambda_0;\lambda_1)$ and $\chi[1](\lambda_0;\lambda_1)$ are solutions of the system (3) with Φ transforming as

$$\Phi(\lambda_0) \to \Phi[1](\lambda_0;\lambda_1) = \Phi(\lambda_0) + \mathrm{i}\ln\frac{\psi_1(\lambda_1)}{\phi_1(\lambda_1)}.,\tag{6}$$

where $\Phi[1](\lambda_0;\lambda_1)$ is a new solution of equation (2) provided that the superfields ψ, ϕ and χ transform according to the transformations given in (5).

Similarly, one can have the following transformations:

$$\mathrm{e}^{\mathrm{i}\Phi(\lambda_0)} \to \mathrm{e}^{\mathrm{i}\Phi[1](\lambda_0;\lambda_1)} = \mathrm{e}^{\mathrm{i}\Phi(\lambda_0)}\frac{\phi_1(\lambda_1)}{\psi_1(\lambda_1)},\tag{7}$$

$$D_-\Phi(\lambda_0) \to D_-\Phi[1](\lambda_0;\lambda_1) = -D_-\Phi(\lambda_0) + \sqrt{\lambda_1}\left[\frac{\chi_1(\lambda_1)}{\psi_1(\lambda_1)} + \frac{\chi_1(\lambda_1)}{\phi_1(\lambda_1)}\right],\tag{8}$$

$$D_+\Phi(\lambda_0) \to D_+\Phi[1](\lambda_0;\lambda_1) = D_+\Phi(\lambda_0) - \frac{1}{2\sqrt{\lambda_1}}\left[\mathrm{e}^{\mathrm{i}\Phi(\lambda_0)}\frac{\chi_1(\lambda_1)}{\psi_1(\lambda_1)} + \mathrm{e}^{-\mathrm{i}\Phi(\lambda_0)}\frac{\chi_1(\lambda_1)}{\phi_1(\lambda_1)}\right].\tag{9}$$

In these transformations, $\Phi[1]$ is another solution of the super sine-Gordon equation (2) with $\lambda = \lambda_0$ generated through the Darboux transformation. In fact the solution $\Phi[1]$ is expressed in terms of solution $\Phi(\lambda_0)$ and particular solutions $\psi_1(\lambda_1)$ and $\phi_1(\lambda_1)$ of the linear system (3) with $\lambda = \lambda_1$.

To establish a connection between the Darboux transformation and Bäcklund transformation of the super sine-Gordon equation (2), we write $\Gamma = \frac{\psi_1}{\phi_1}$ and $k = \frac{\chi_1}{\psi_1}$ and express equation (8) in terms of Γ and k as

$$D_-\Phi[1] = -D_-\Phi + \sqrt{\lambda_1}\,[k + k\Gamma]\,.$$

Now eliminating Γ and k by a transformation $\Gamma = \exp\mathrm{i}(\Phi - \Phi[1])$ and $f = k\sqrt{\Gamma}$, we have

$$D_-(\Phi + \Phi[1]) = 2\sqrt{\lambda_1}\,f\cos\left(\frac{\Phi - \Phi[1]}{2}\right),\tag{10}$$

[4] We can also have a gauge equivalent description of the linearization and the Lax operator solves the isospectral problem for the gauge transformed eigenfunctions.

where f is another fermionic superfield. In the same manner we can find the other part of the Bäcklund transformation, from equation (9),

$$D_{+}(\Phi - \Phi[1]) = \frac{1}{\sqrt{\lambda_1}} f \cos\left(\frac{\Phi + \Phi[1]}{2}\right). \tag{11}$$

The superfield f is subjected to the following conditions:

$$D_{+} f = \frac{\mathrm{i}}{\sqrt{\lambda_1}} \sin\left(\frac{\Phi + \Phi[1]}{2}\right), \tag{12}$$

$$D_{-} f = -2\mathrm{i}L\sqrt{\lambda_1} \sin\left(\frac{\Phi - \Phi[1]}{2}\right). \tag{13}$$

Equations (10)–(13) are exactly the Bäcklund transformation of the super sine-Gordon equation already obtained in [5]. The super soliton solution can be obtained by taking $\Phi = 0$ in (6) to give a solution of the linear problem (3) as

$$\begin{aligned} \psi(x^{\pm}, \theta^{\pm}) &= A_0^{+}(\theta^{+}, \theta^{-}) \exp(\eta) + A_0^{-}(\theta^{+}, \theta^{-}) \exp(-\eta), \\ \phi(x^{\pm}, \theta^{\pm}) &= A_0^{+}(\theta^{+}, \theta^{-}) \exp(\eta) - A_0^{-}(\theta^{+}, \theta^{-}) \exp(-\eta), \\ \chi(x^{\pm}, \theta^{\pm}) &= A_1(\theta^{+}, \theta^{-}) \exp(\eta), \end{aligned}$$

where $A_0^{\pm}(\theta^{+}, \theta^{-})$ are even supernumbers and $A_1(\theta^{+}, \theta^{-})$ is an odd supernumber and $\eta = \frac{1}{4\lambda}x^{+} + \lambda x^{-}$. By substituting this, one can obtain a single super soliton solution as

$$\Phi[1] = \mathrm{i} \ln \left[\frac{1 + \mathcal{A}_0(\theta^{+}, \theta^{-}) \exp(-2\eta)}{1 - \mathcal{A}_0(\theta^{+}, \theta^{-}) \exp(-2\eta)}\right],$$

where $\mathcal{A}_0(\theta^{+}, \theta^{-})$ is some even supernumber. The two super soliton solution can be obtained from the two-fold Darboux transformation as

$$\begin{aligned} \Phi[2] &= \Phi[1] + \mathrm{i} \ln \frac{\psi_1[1]}{\phi_1[1]} \\ &= \mathrm{i} \ln \left[\frac{\Delta_{12}^{1}[2] + \mathrm{i}\sqrt{\lambda_1\lambda_2} X_{12}}{\Delta_{12}^{2}[2] + \mathrm{i}\sqrt{\lambda_1\lambda_2} X_{12}}\right], \end{aligned}$$

where

$$\Delta_{12}^{1}[2] = \det\begin{pmatrix} \lambda_1\phi_1 & \lambda_2\phi_2 \\ \psi_1 & \psi_2 \end{pmatrix},$$

$$\Delta_{12}^{2}[2] = \det\begin{pmatrix} \lambda_1\psi_1 & \lambda_2\psi_2 \\ \phi_1 & \phi_2 \end{pmatrix},$$

$$X_{12} = \chi_1\chi_2.$$

Similarly, the three super soliton solution as obtained by using the three-fold Darboux transformation is given by

$$\begin{aligned} \Phi[3] &= \Phi[2] + \mathrm{i} \ln \frac{\psi_1[2]}{\phi_1[2]} \\ &= \mathrm{i} \ln \left[\frac{\begin{array}{c} \Delta_{123}^{1}[3] + \mathrm{i}\sqrt{\lambda_2\lambda_3}(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)\Delta_1^1[1]X_{23} \\ + \mathrm{i}\sqrt{\lambda_1\lambda_3}(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)\Delta_2^1[1]X_{13} \\ + \mathrm{i}\sqrt{\lambda_1\lambda_2}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\Delta_3^1[1]X_{12} \end{array}}{\begin{array}{c} \Delta_{123}^{2}[3] + \mathrm{i}\sqrt{\lambda_2\lambda_3}(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)\Delta_1^2[1]X_{23} \\ + \mathrm{i}\sqrt{\lambda_1\lambda_3}(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)\Delta_2^2[1]X_{13} \\ + \mathrm{i}\sqrt{\lambda_1\lambda_2}(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)\Delta_3^2[1]X_{12} \end{array}}\right], \end{aligned}$$

where the determinants in the above expressions are given by

$$\Delta^1_{123}[3] = \det\begin{pmatrix} \lambda_1^2\psi_1 & \lambda_2^2\psi_2 & \lambda_3^2\psi_3 \\ \lambda_1\phi_1 & \lambda_2\phi_2 & \lambda_3\phi_3 \\ \psi_1 & \psi_2 & \psi_3 \end{pmatrix}, \qquad \Delta^2_{123}[3] = \det\begin{pmatrix} \lambda_1^2\phi_1 & \lambda_2^2\phi_2 & \lambda_3^2\phi_3 \\ \lambda_1\psi_1 & \lambda_2\psi_2 & \lambda_3\psi_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix},$$

$$\Delta^1_l[1] = \psi_l, \qquad \Delta^2_l[1] = \phi_l, \qquad X_{ij} = \chi_i\chi_j.$$

The super N-soliton solution is obtained by iteration to the N-fold Darboux transformation

$$\begin{aligned}\Phi[N] = \mathrm{i}\ln\Bigg[\Bigg\{ & \sum_{\substack{\text{all possible}\\ \text{pairings } M}} \sum_{k_1<k_2<\cdots<k_{2M}} (\mathrm{i})^M \sqrt{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_{2M}}}\, P_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}(N-2M) \\ & \times\Bigg[\Delta^1_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M] + \frac{(\mathrm{i})^M}{M!}\sqrt{\lambda_1\cdots\hat{\lambda}_{k_1}\hat{\lambda}_{k_2}\cdots\hat{\lambda}_{k_{2M}}\cdots\lambda_N} \\ & \times X_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M]\Bigg] X_{k_1k_2\cdots k_{2M}}[2M]\Bigg\}\Bigg/ \\ & \Bigg\{ \sum_{\substack{\text{all possible}\\ \text{pairings } M}} \sum_{k_1<k_2<\cdots<k_{2M}} (\mathrm{i})^M \sqrt{\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_{2M}}}\, P_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}(N-2M) \\ & \times\Bigg[\Delta^2_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M] + \frac{(\mathrm{i})^M}{M!}\sqrt{\lambda_1\cdots\hat{\lambda}_{k_1}\hat{\lambda}_{k_2}\cdots\hat{\lambda}_{k_{2M}}\cdots\lambda_N} \\ & \times X_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M]\Bigg] X_{k_1k_2\cdots k_{2M}}[2M]\Bigg\}\Bigg], \end{aligned} \tag{14}$$

where $0 \leqslant M \leqslant \frac{N}{2}$. In formula (14), the caret above a term or an index means that it is to be omitted from the product and the sum runs over all permutations. If $k_1k_2\cdots k_{2M}$ are omitted indices and $12\cdots N-2M$ are unomitted indices, then the polynomial $P_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}(N-2M)$ is given by

$$P_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}(N-2M) = (\lambda_1+\lambda_{k_1})(\lambda_1+\lambda_{k_2})\cdots(\lambda_1+\lambda_{k_{2M}})$$
$$(\lambda_2+\lambda_{k_1})(\lambda_2+\lambda_{k_2})\cdots(\lambda_2+\lambda_{k_{2M}})\cdots(\lambda_N+\lambda_{k_1})(\lambda_N+\lambda_{k_2})\cdots(\lambda_N+\lambda_{k_{2M}})$$
$$\text{with}\quad P_{12\cdots N}(N) = P_{\hat{1}\hat{2}\cdots\hat{N}}(0) = 1.$$

The other components in formula (14) are defined as

$$\Delta^1_{12\cdots N}[N] = \det\begin{pmatrix} \lambda_1^{N-1}\psi_1 & \lambda_2^{N-1}\psi_2 & \lambda_3^{N-1}\psi_3 & \cdots & \lambda_N^{N-1}\psi_N \\ \lambda_1^{N-2}\phi_1 & \lambda_2^{N-2}\phi_2 & \lambda_3^{N-2}\phi_3 & \cdots & \lambda_N^{N-2}\phi_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1\phi_1 & \lambda_2\phi_2 & \lambda_3\phi_3 & \cdots & \lambda_N\phi_N \\ \psi_1 & \psi_2 & \psi_3 & \cdots & \psi_N \end{pmatrix},$$

$$\Delta^2_{12\cdots N}[N] = \det\begin{pmatrix} \lambda_1^{N-1}\phi_1 & \lambda_2^{N-1}\phi_2 & \lambda_3^{N-1}\phi_3 & \cdots & \lambda_N^{N-1}\phi_N \\ \lambda_1^{N-2}\psi_1 & \lambda_2^{N-2}\psi_2 & \lambda_3^{N-2}\psi_3 & \cdots & \lambda_N^{N-2}\psi_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1\psi_1 & \lambda_2\psi_2 & \lambda_3\psi_3 & \cdots & \lambda_N\psi_N \\ \phi_1 & \phi_2 & \phi_3 & \cdots & \phi_N \end{pmatrix},$$

$X_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M] = \chi_1\chi_2\cdots\hat{\chi}_{k_1}\hat{\chi}_{k_2}\cdots\hat{\chi}_{k_{2M}}\cdots\chi_N$, with $[N-2M]$ even,
$X_{k_1k_2\cdots k_{2M}}[2M] = \chi_{k_1}\chi_{k_2}\cdots\chi_{k_{2M}}$.

In formula (14), the term $X_{12\cdots\hat{k}_1\hat{k}_2\cdots\hat{k}_{2M}\cdots N}[N-2M]$ will only appear when N is even.

In summary, we have investigated the Darboux transformation for the super sine-Gordon equation and obtained explicit transformations on the sine-Gordon superfield as well as on the superfields of the linear system associated with the equation. We have also established a connection between the super Darboux transformation and the super Bäcklund transformation of the equation. At the end of the paper, we have used the Darboux transformation to obtain N super soliton solutions of the equation and the solution appears in the form of the product of Wronskian determinants and even number of fermionic superfields.

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